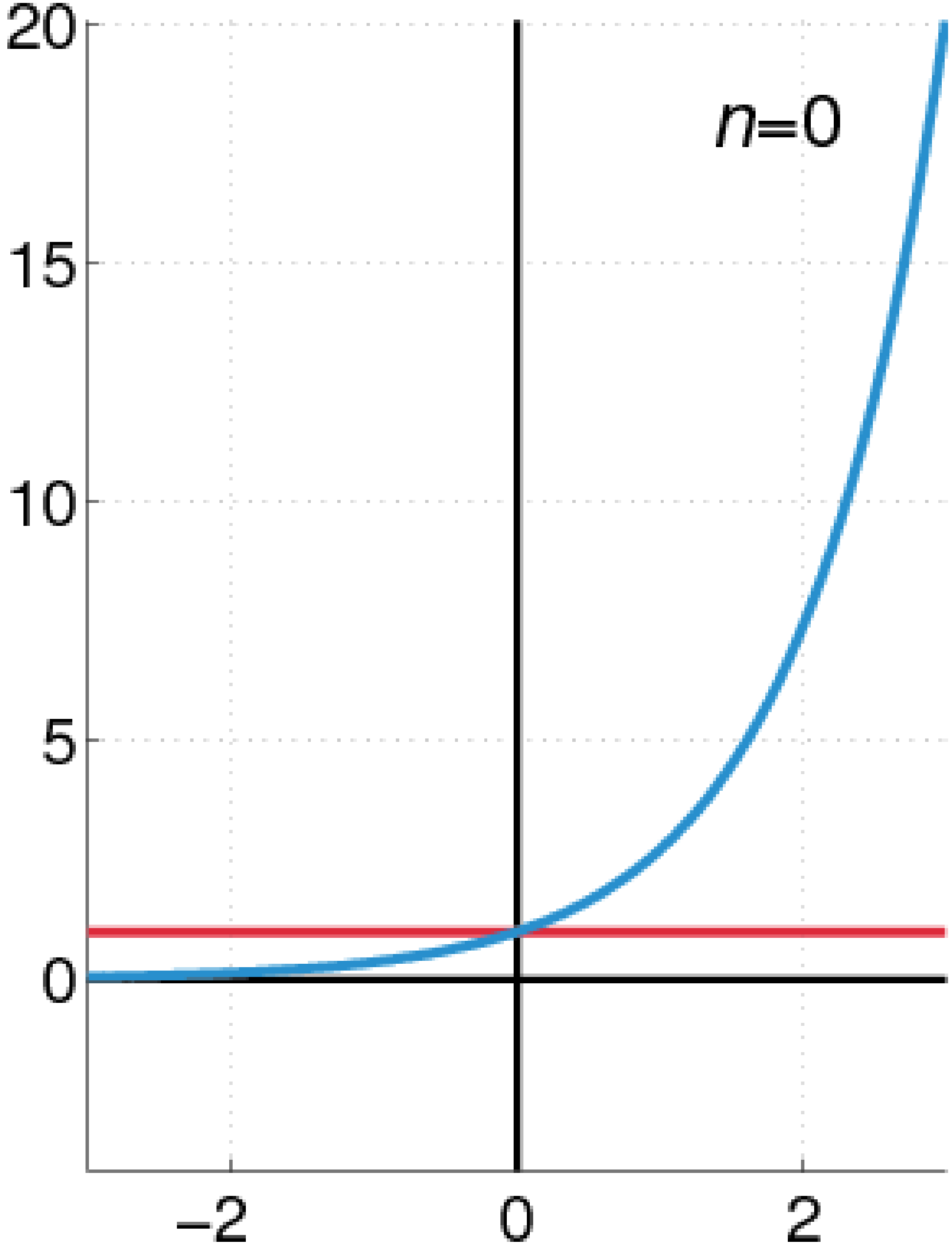


20
15
10
5
0
n=0
−2
0
2

We've learned that it can be very hard to tell what an always increasing function *(like an improper integral)* or sequence (like a sequence of partial sums) does in the long term.

| f (x) | $\int^{\infty}$ f (x) dx ; antiderivative of f(x) |
|---|---|
| f(x) $\geq$ 0 | Antiderivative is increasing |
| Decreasing | Antiderivative is concave down |
| Limit of f(x) goes to zero | Antiderivative is getting flatter/ increasing more slowly; its slope is tending to 0 |

There are increasing concave down functions that diverge e.g. y=ln(x)
There are also increasing concave down function with horizontal asymptotes e.g y= 1 – 1/x
The antiderivative (improper integral) could be like either one. It depends on the speed at which the slope goes to zero, which is exactly the speed that f (x) goes to zero.

# Series Solving Tip:

Try to figure out in your head whether the series converges or diverges before you begin;  Ask yourself:

-What is the relative strength of the numerator and denominator?
-Is the denominator more than “n” stronger?
-Which terms here matter when n is really, really big?

Your justification will probably help you identify which test to use, and let you know if the answer you get is reasonable.

But BE CAREFUL: such explanations are not valid answers on an exam. One could incorrectly but easily say that “n is MUCH bigger than 1, so the sum of 1/n converges”.  You have to actually use mathematics (i.e. limits, L’Hospital’s, and tests) to *prove* what you are saying.  A lot of problems come back to knowing the convergence of a p-series.

| | |
|---|---|
| *Nth term* | Terms look suspicious ; may not be going to zero |
| *Integral* | That's looks like something I can integrate! |
| *Comparison* | There are some terms that are pretty irrelevant when n is very very large that I'd like to be rid of |
| *Limit Comparison* | Series behaves like a simpler series (especially a p-series) in the limit |
| *Ratio* | Consecutive terms would cancel a lot; n!, x^n, or both |
| *Root* | Lots of powers to the n, but n appears elsewhere in ways that make the ratio test hard. |
| *Absolute Convergence* | Ignoring minus signs gives me a familiar series; There are negative terms, but not alternating |
| *Alternating Series* | Any alternating Series |

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

General Power Series. "Infinite Polynomial"
Converges for some x , which ones depends on the values of the sequence of constants $c_n$ . Converges to some unknown function in said interval

$$\sum_{n=0}^{\infty} x^n$$

Geometric-like power series.
Converges if x is in (-1,1).
Converges to the function f (x) = 1/(1-x)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Mystery function; limit of $P_n$ for f (x) = $e^x$
Converges Everywhere
Converges to the function f (x) = $e^x$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

Limit of the polynomial approximations Pn of f(x)
Interval of convergence depends on the coefficients $f^{(n)}(0)$, and thus on the function f(x)
Converges to the function f (x)

# Power Series Convergence???

How can we tell if and where a power series converges?

Use our convergence tests- the answer that they give (converge/diverge) will usually depend on what x is. The ratio test is commonly best, but you have to check the values of x where the ratio test yields 1 as special cases.

Power series will always converge in an **interval centered on a.**
**$|x-a| < R$** for some R (note: R can be 0 or infinite)
Which makes sense, since each term is multiplied by another copy of x-a, it's size will directly affect the ratios of the terms.

The endpoints of this interval will always be special cases to consider. Their convergence determines whether you should use round or square brackets. $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R]$

# Useful Things We Can do with Power Series

Term by term differentiation:

If f(x) = $\sum_{n=0}^{\infty} c_n (x-a)^n$

Then f'(x) = $\sum_{n=0}^{\infty} n \cdot c_n (x-a)^{n-1} = \sum_{n=0}^{\infty} \frac{n \cdot c_n}{x-a} \cdot (x-a)^n$

Term by term integration:

F(x) = $\sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} = \sum_{n=0}^{\infty} \frac{c_n (x-a)}{n+1} (x-a)^n$

Note: Neither of these changes the open interval of convergence

# Particularly Awesome Power Series

MacLaurin series generated by f
(Taylor series generated by f at x=0)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

Taylor series generated by f at x=a
(Agrees with all derivatives of f at a)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$$

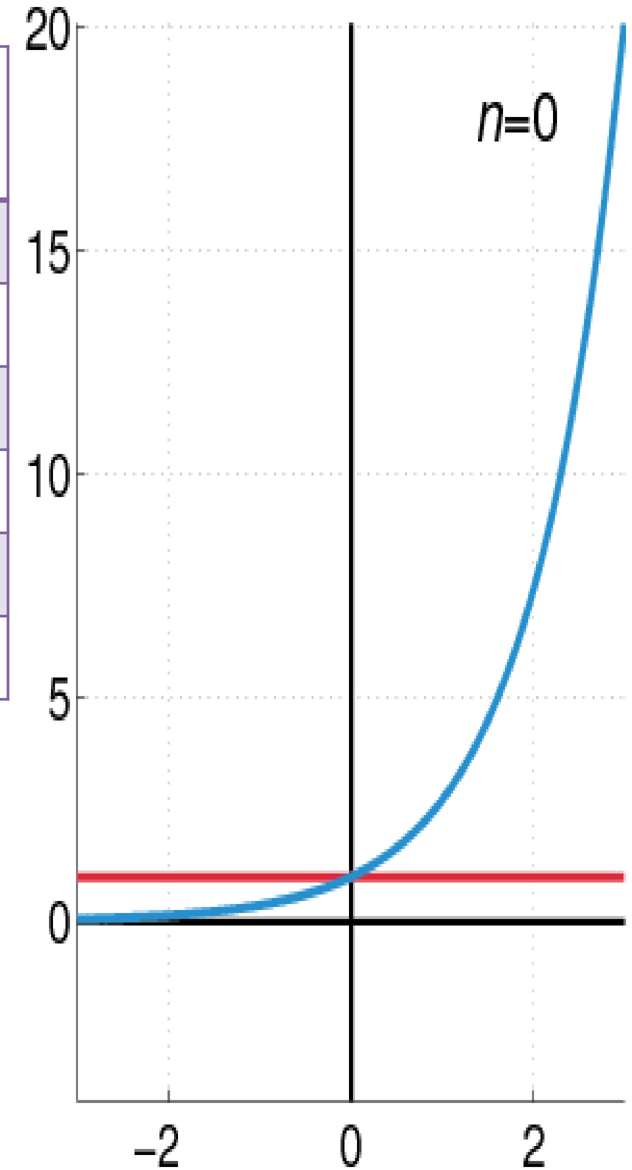
$P_n$ = Taylor polynomial of order n, a finite polynomial approximating f
= The unique polynomial of up to degree n that agrees with the value and first n derivatives of f(x) at some given x=a

Each $P_n$ is the nth partial sum of the taylor series. Limit $P_n$ = Taylor series

Ex: From weekend activity, the MacLaurin series of $e^x$ is: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Let's find the MacLaurin series for $e^x$: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$

| n | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $P_n$ |
|---|---|---|---|
| 0 | $e^x$ | 1 | 1 |
| 1 | $e^x$ | 1 | $1 + x$ |
| 2 | $e^x$ | 1 | $1 + x + x^2/2!$ |
| 3 | $e^x$ | 1 | $1 + x + x^2/2! + x^3/3!$ |
| 4 | $e^x$ | 1 | $1 + x + x^2/2! + x^3/3! + x^4/4!$ |
| 5 | $e^x$ | 1 | $1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5!$ |




Every $f^{(n)}(0)$ equals 1

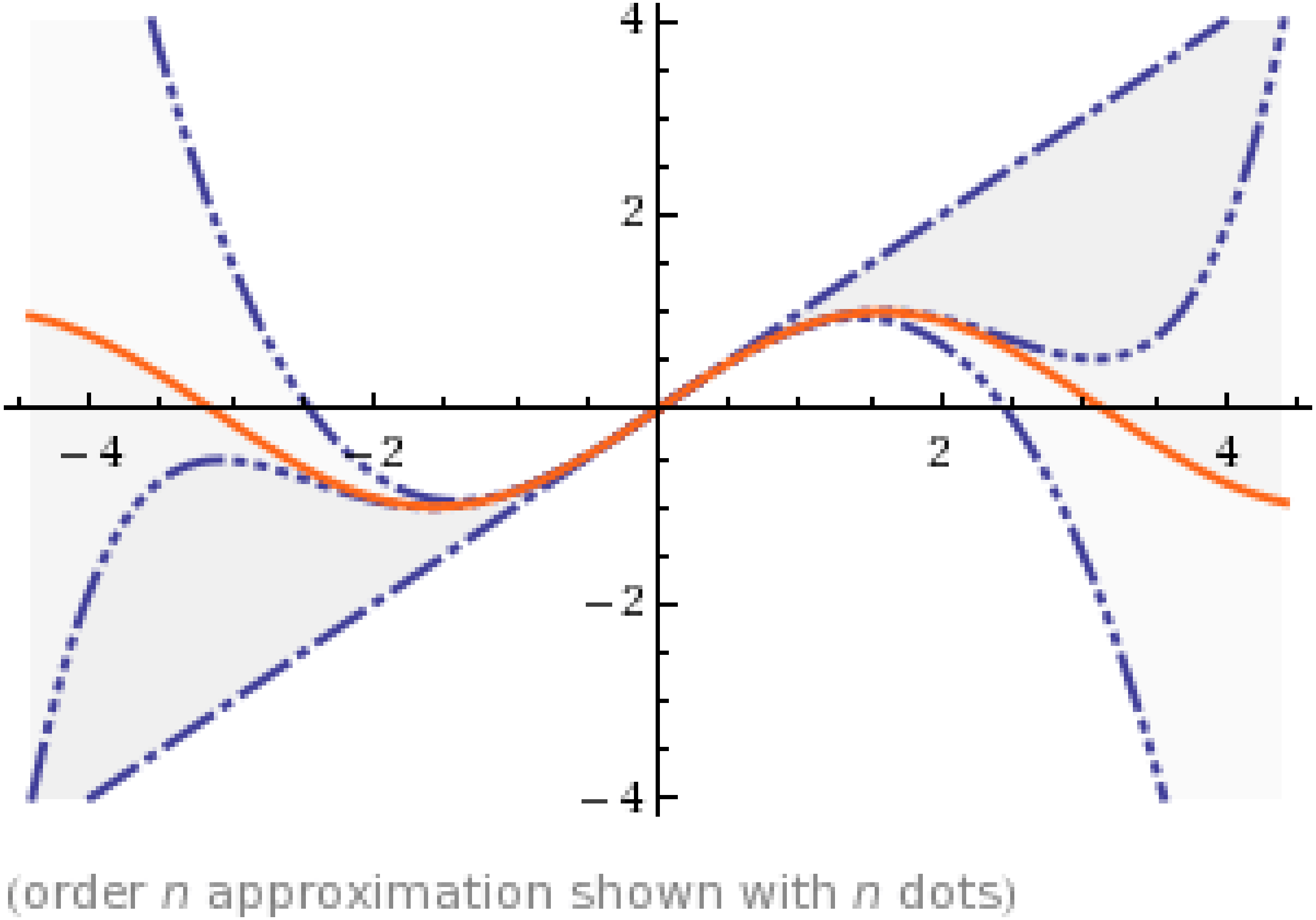
We get a lovely power series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Let's find the MacLaurin series for sin(x): $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$

| n | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $P_n$ |
|---|---|---|---|
| 0 | sin(x) | 0 | 0 |
| 1 | cos(x) | 1 | x |
| 2 | -sin(x) | 0 | x + 0 |
| 3 | -cos(x) | -1 | $x - x^3/3!$ |
| 4 | sin(x) | 0 | $x - x^3/3! + 0$ |
| 5 | cos(x) | 1 | $x - x^3/3! + x^5/5!$ |
| 6 | -sin(x) | 0 | $x - x^3/3! + x^5/5! + 0$ |
| 7 | -cos(x) | -1 | $x - x^3/3! + x^5/5! - x^7/7!$ |

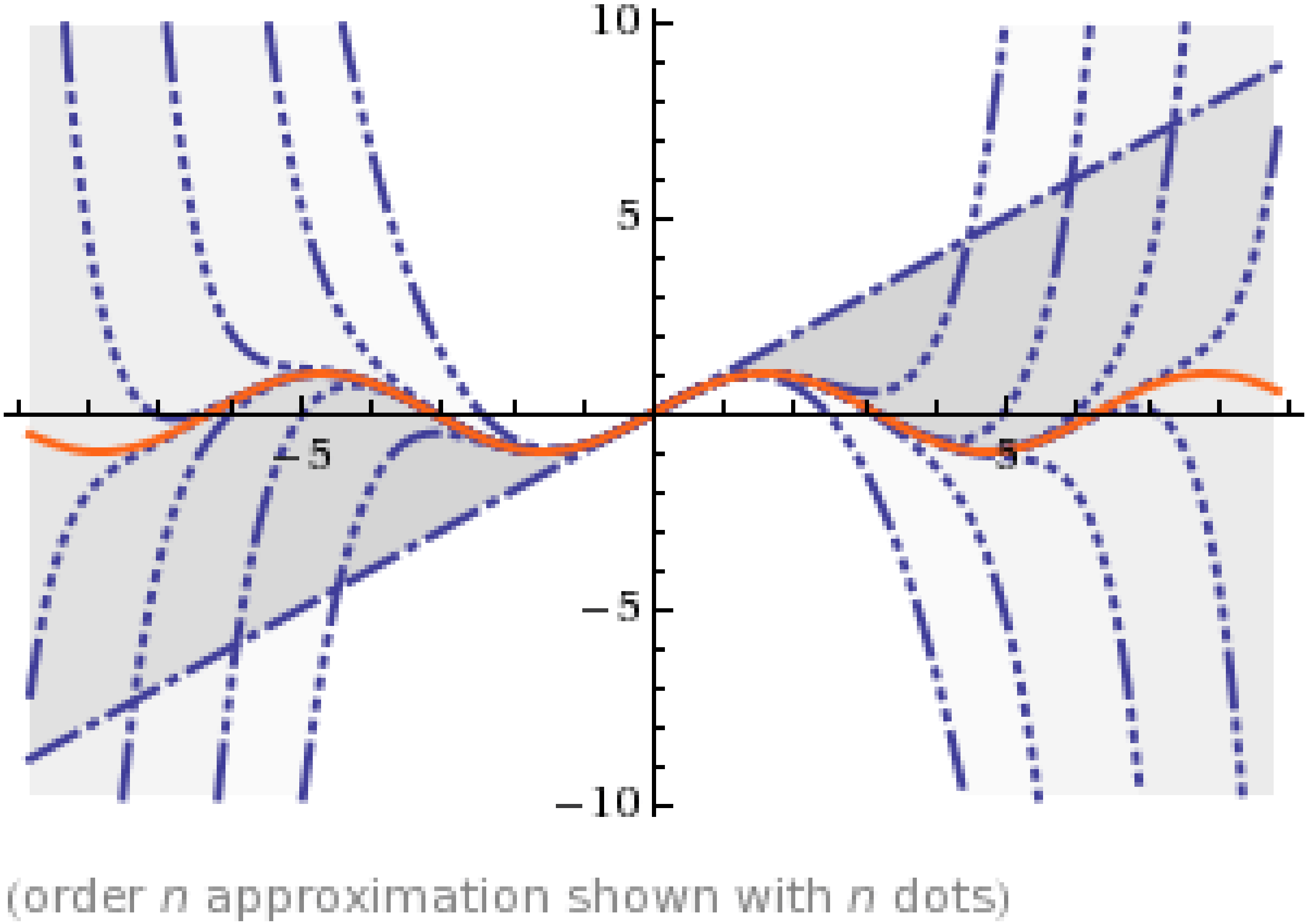
So we only want terms in the odd degrees, with coefficients alternating between +1 and -1

$$\sum_{k=0}^{\infty} \frac{(-1)^k \, x^{1+2k}}{(1+2k)!}$$



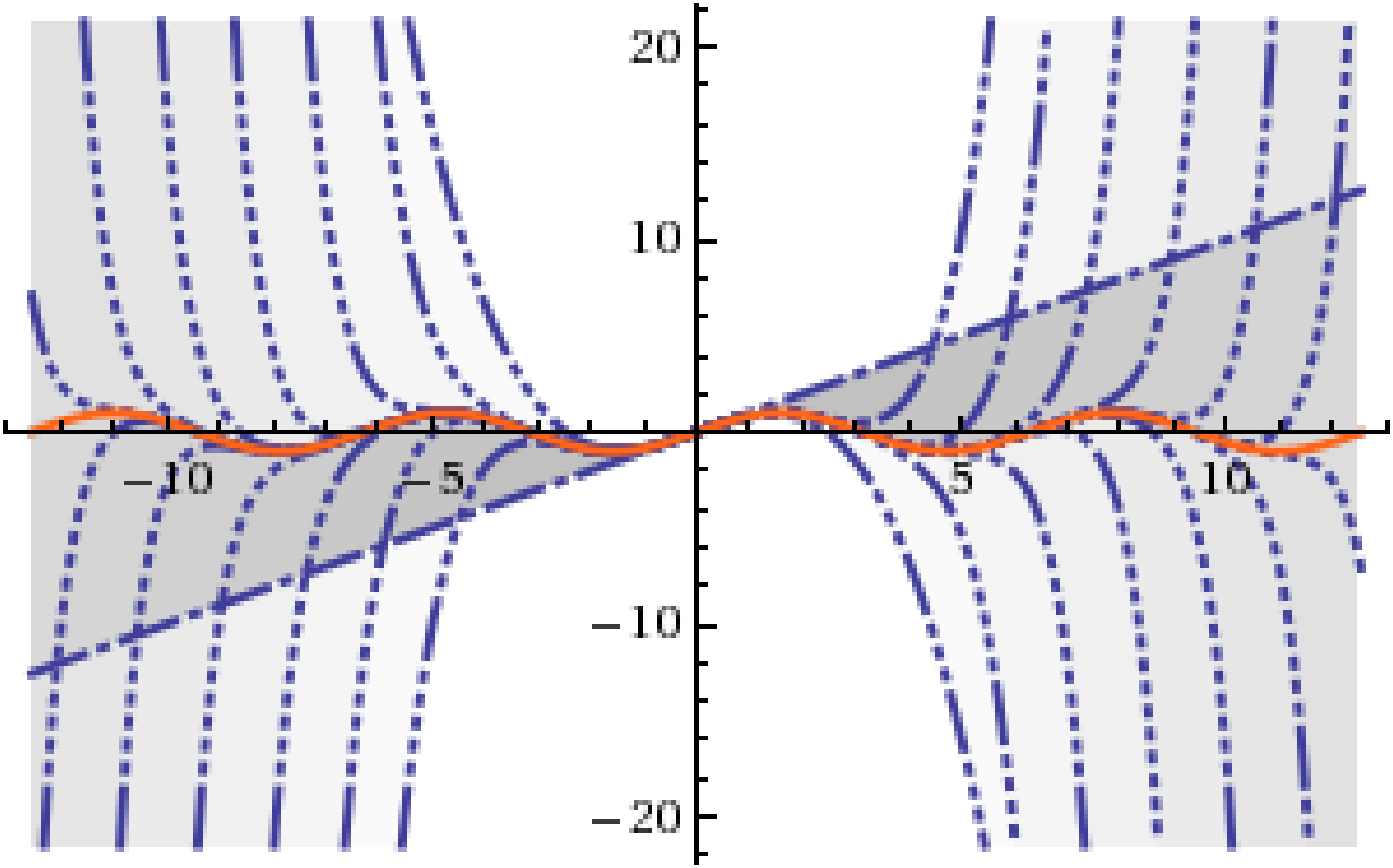

Approximations of y=sin(x) up to $P_5$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \, x^{1+2k}}{(1+2k)!}$$




Approximations of y=sin(x) up to $P_{17}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \, x^{1+2k}}{(1+2k)!}$$




(order $n$ approximation shown with $n$ dots)

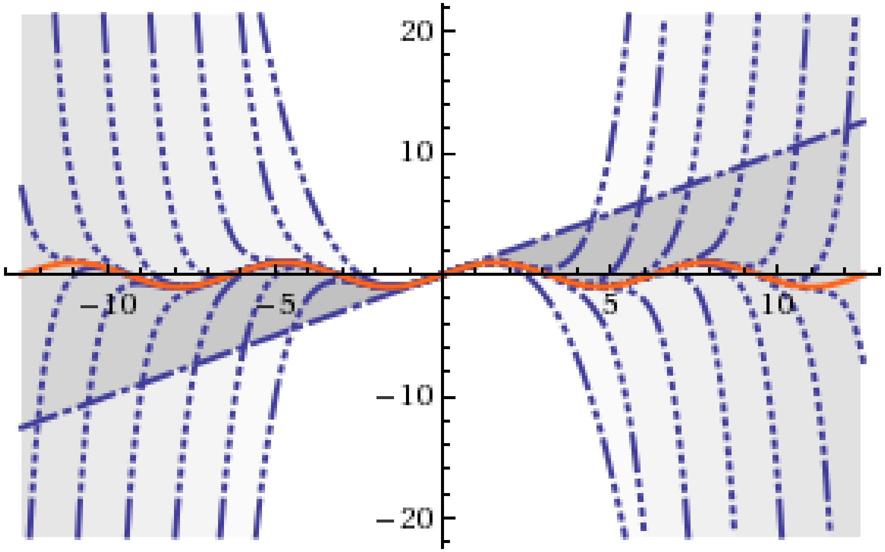
Approximations of y=sin(x) up to $P_{27}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{(1+2k)!}$$

# Convergence

It looks like if we let the polynomials increase in degree forever, the approximations will keep getting closer to sin(x).

This is the same as saying that the Taylor series converges everywhere and




(order $n$ approximation shown with $n$ dots)

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{1+2k}}{(1+2k)!}$$

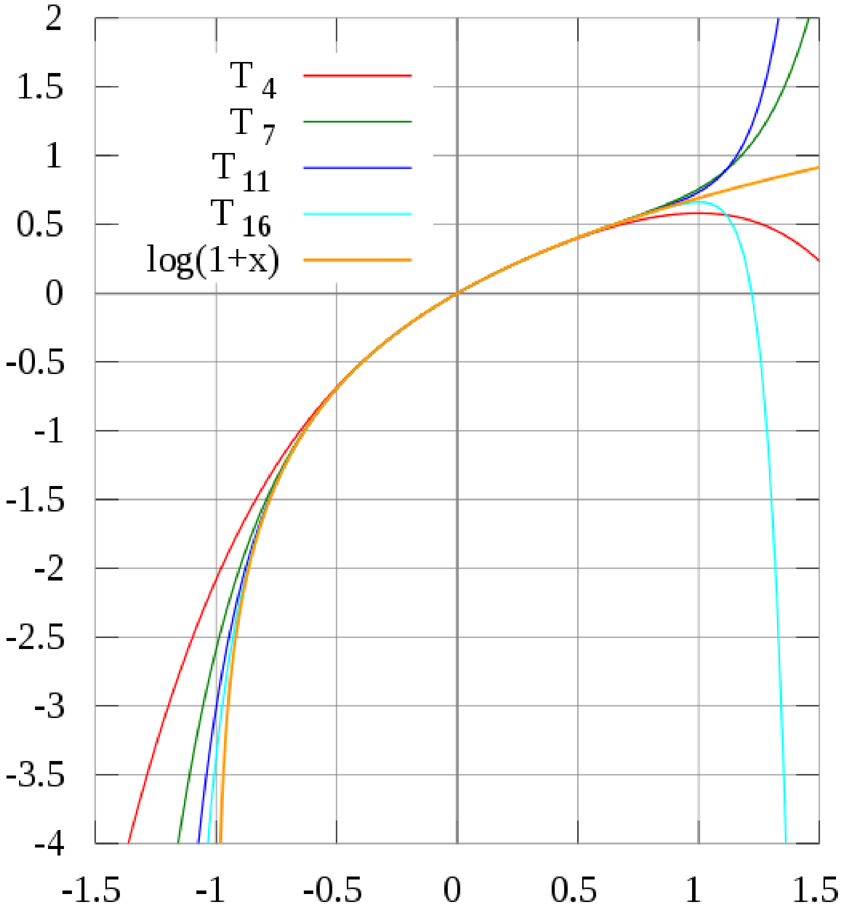
But why should this be true?

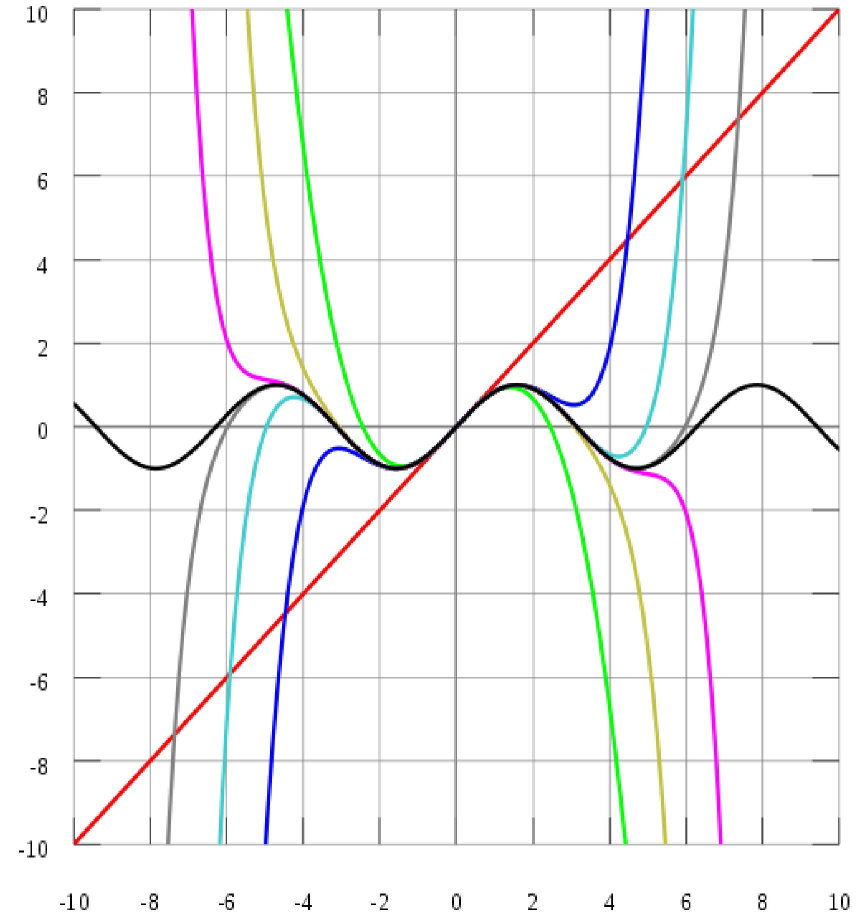
The Taylor series is absolutely convergent for every x by the ratio test:

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|x|^{1+2(k+1)}}{(1+2(k+1))!} \frac{(1+2k)!}{(-1)^k\, |x|^{1+2k}}$$

$$= \lim_{k\to\infty} \frac{|x|^2}{(1+2k+2)(1+2k+1)} = 0$$

# Some Pretty Taylor Polynomials



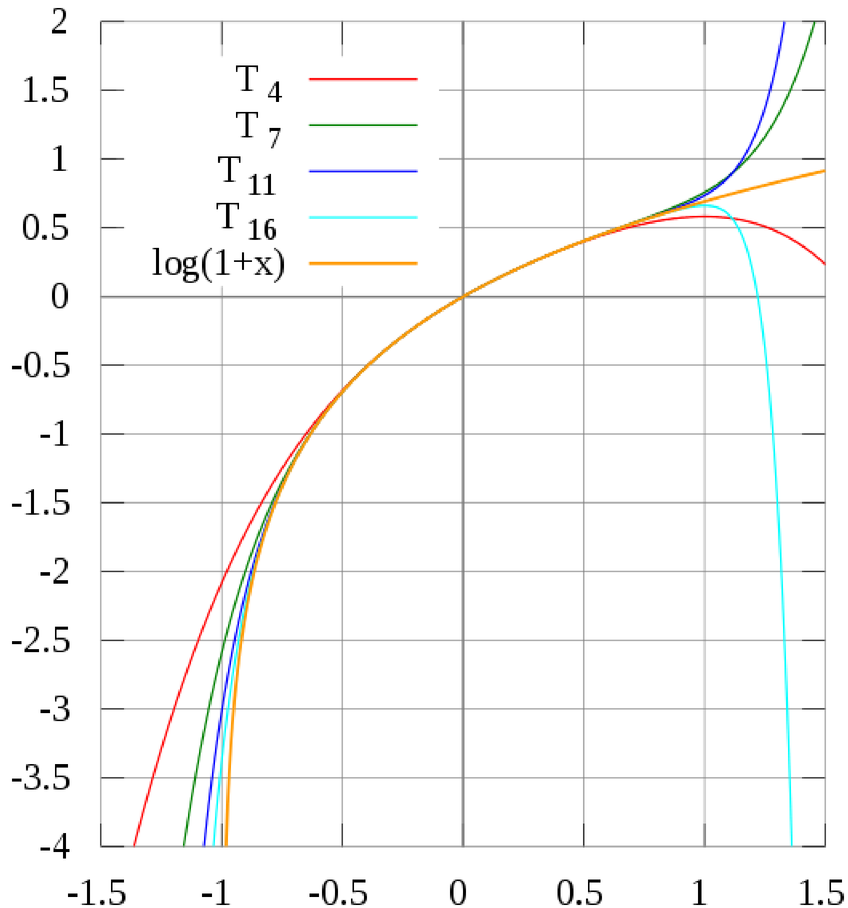



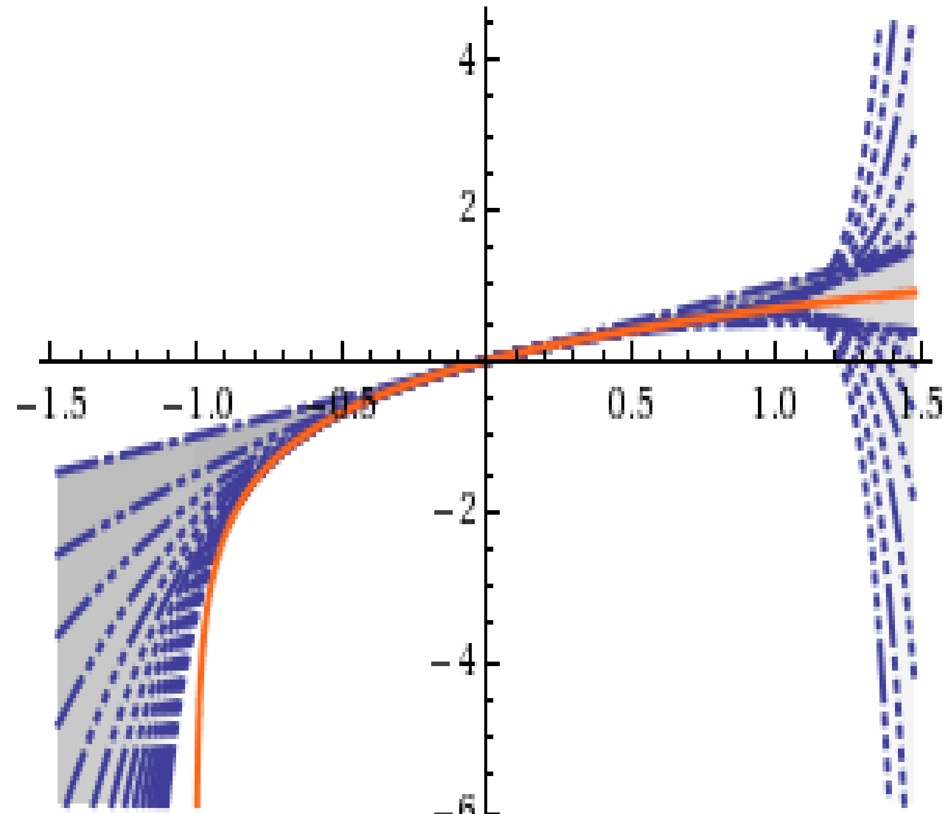

Approximations of y = ln(1+x), actual curve shown in orange

Approximations of y = sin(x), actual curve shown in black

Let's find the MacLaurin Series of ln(1+x). Recall: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$

| N | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $P_n$ |
|---|---|---|---|
| 0 | ln(1+x) | 0 | 0 |
| 1 | $(1+x)^{-1}$ | 1 | x |
| 2 | $-(1+x)^{-2}$ | -1 | $x - x^2/2$ |
| 3 | $2(1+x)^{-3}$ | 2 | $x - x^2/2 + x^3/3$ |
| 4 | $-6(1+x)^{-4}$ | -6 | $x - x^2/2 + x^3/3 - x^4/4$ |
| 5 | $24(1+x)^{-5}$ | 24 | $x - x^2/2 + x^3/3 - x^4/4 + x^5/5$ |
| 6 | $-120(1+x)^{-6}$ | -120 | $x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6$ |
| | | | |
| n | $(-1)^{n}(n-1)!(1+x)^{-n}$ | $(-1)^{n}(n-1)!$ | $\Sigma_0^{k=n} ((-1)^{k} / k) \cdot x^k$ |




(order *n* approximation shown with *n* dots)

Something odd is happening as the approximations get close to 1. The *better* approximations seem to be rushing down and away from the curve rather quickly.

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{k}$$

Only between -1 and 1?
Specifically, for x in ( -1, 1 ]